

计算陆架风海流的一种模式*

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研究大陆架上的风海流,必须同时考虑海面坡度和海底地形(变水深)的影响。要想获得分析解是比较困难的。

作为对大陆架地形的一级近似,本文考虑线性变化的倾斜海底,岸边水深为零。并假定海岸线平行于地球经度线,考虑柯氏参数随纬度的变化,把这个 β 效应项—— βy 作为小参数,同时假定定常二维风场随纬度也是按小参数 βy 变化着。

将流速和海面坡度都按小参数 βy 展开,我们就得到确定流速和海面坡度的一系列方程组。在极坐标下求解这些方程,以 Bessel 函数作为基本函数,本文给出了构造级数形式的分析解的一套有效方法,并给予了严格证明。为了保持解在岸边尖点处的正则性质,本文采用的是分段解,然后再在离岸某个距离处将两部分解匹配起来。从而就得到了适用于整个海域的分析解,这样我们就克服了 Hill 和 Johnson 在他们的“大陆架上的沿岸流和上升流的三维模式”^[1]一文中的没能克服的困难,从而有可能更细致地研究大陆架上风海流的结构。

一、问题的数学提法

本文的目的是研究具有线性倾斜海底的半无界海域上的定常风海流问题。设海岸线平行于地球经度线。取右手坐标系 (O, X, Y, Z) , 如图 1 所示。 Y 轴与岸边重合, X 轴在平均海平面上指向海域, Z 轴铅直向上为正。

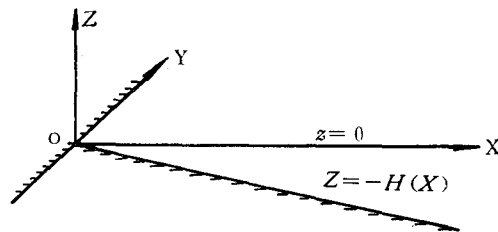


图1 坐标的选取

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$Z=0$ 为平均海平面, $Z=-H(X)=-\Theta X$ 为海底, 这里 $\Theta=\text{const}$ 为海底坡度。海面升高为 $\zeta(X, Y)$ 。设海水均匀 $\rho = \text{const}$, 考虑柯氏参数随纬度的变化, 即考虑 β 效应。这样柯氏参数 $F = F_0 + \beta Y$, 这里 $F_0 = 2\Omega \sin \phi_0$, $\beta = \frac{2\Omega}{a} \cos \phi_0$, ϕ_0 为地理纬度, Ω 为地球自转角速度, $a = 6370$ 公里为地球平均半径。并假定垂向和侧向(水平)涡动粘滞系数 A_v, A_h 为常数。

这样, 在忽略非线性效应的情况下, 我们得到的定常流动的运动方程式为

$$\begin{aligned} \frac{A_h}{\rho} \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) + \frac{A_v}{\rho} \frac{\partial^2 U}{\partial Z^2} + FV &= g \frac{\partial \zeta}{\partial X}, \\ \frac{A_h}{\rho} \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \frac{A_v}{\rho} \frac{\partial^2 V}{\partial Z^2} - FU &= g \frac{\partial \zeta}{\partial Y}, \end{aligned} \quad (1)$$

连续性方程为

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0, \quad (2)$$

边界条件为

$$\begin{aligned} A_v \frac{\partial U}{\partial Z} &= \sigma_1(X, Y), \quad A_v \frac{\partial V}{\partial Z} = \sigma_2(X, Y), \quad \text{当 } Z = 0 \\ U = W = V, & \quad \text{当 } Z = -H(X) \\ U = V = 0, & \quad \text{当 } X = 0 \\ U + iV \rightarrow (U + iV)_E &= \frac{\sigma_{1,0} + i\sigma_{2,0}}{\sqrt{2\rho A_v \Omega \sin \phi_0}} e^{-(\pi + i\pi) \frac{|Z|}{D_v} - i \frac{\pi}{4}}, \quad \text{当 } X \rightarrow \infty \end{aligned} \quad (3)$$

这里 U, V, W 分别为 X, Y, Z 三个方向上的流速分量。 $\sigma_1(X, Y), \sigma_2(X, Y)$ 分别为海面风应力在 X, Y 方向上的分量, 并且当 $X \rightarrow \infty$ 时风应力分布不依赖于 X , 即 $\sigma_1(X, Y) \rightarrow \sigma_{1,0}(Y), \sigma_2(X, Y) \rightarrow \sigma_{2,0}(Y)$ 。这里 $(U + iV)_E$ 为风场 $\sigma_{1,0} + i\sigma_{2,0}$ 作用下无限深海的 Ekman 漂流。

为方便计, 作如下无量纲化^{[2], [1]}。取

$$\begin{aligned} x &= \frac{X}{D_h}, \quad y = \frac{Y}{D_h}, \quad z = \frac{Z}{D_v}, \quad u = \frac{U}{\sqrt{gD_v}}, \quad v = \frac{V}{\sqrt{gD_v}}, \quad w = \frac{kW}{\sqrt{gD_v}}, \\ d_1(x, y) &= \frac{\partial \zeta}{\partial X} / \zeta'_0, \quad d_2(x, y) = \frac{\partial \zeta}{\partial Y} / \zeta'_0, \quad \tau_1(x, y) = \sigma_1(x, y) / A_v \sqrt{\frac{g}{D_v}}, \\ \tau_2(x, y) &= \sigma_2(x, y) / A_v \sqrt{\frac{g}{D_v}}, \quad f_0 = F_0 / \frac{A_v}{\rho D_v^2} = 2\pi^2, \\ \varepsilon y &= \frac{\beta Y}{F_0} = \frac{2\Omega}{a} \cos \phi_0 Y / 2\Omega \sin \phi_0 = \frac{Y}{a} \text{ctg} \phi_0 = \frac{D_h}{a} \text{ctg} \phi_0 \cdot y, \quad \varepsilon = \frac{D_h}{a} \text{ctg} \phi_0, \\ h(x) &= H(X) / D_v = \frac{\Theta X}{D_h} \cdot \frac{D_h}{D_v} = k\Theta x = \text{tg} \theta_0 \cdot x, \quad \text{其中 } \theta_0 = \text{tg}^{-1} k\Theta \text{ 为 } (x, y, z) \text{ 坐标系中海底的倾斜角。} \end{aligned}$$

1) 这里的无量纲化只是为了使无量纲后的方程形式简单, 没有作量阶估计的目的, 各无量纲不必是 1 的量阶。

这里

$$k^2 = \frac{A_h}{A_D} \left(\text{或 } k = \frac{D_h}{D_v} \right), \quad \zeta'_0 = \frac{A_v}{\rho g^{1/2} D_v^{3/2}},$$

摩擦深度

$$D_v = \pi \sqrt{\frac{A_v}{\rho Q \sin \phi_0}}, \quad \text{摩擦宽度 } D_h = \pi \sqrt{\frac{A_h}{\rho Q \sin \phi_0}}. \quad (4)$$

引进复速度 $W(x, y, z) = u(x, y, z) + iv(x, y, z)$, 复海面坡度 $D(x, y) = d_1(x, y) + id_2(x, y)$ 和复应力 $\tau(x, y) = \tau_1(x, y) + i\tau_2(x, y)$, 则无量纲的方程和边界条件为

$$\begin{cases} \Delta W + \frac{\partial^2 W}{\partial y^2} - if_0(1 + \varepsilon y)W = D(x, y), & \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial W}{\partial z} \Big|_{z=0} = \tau(x, y), \\ W \Big|_{z=0} = W \Big|_{z=-h(x)} = 0, \\ W \rightarrow \tau_0 \frac{e^{i\alpha z}}{i\alpha}, \quad \text{当 } x \rightarrow \infty, \end{cases} \quad (5)$$

其中 $\tau(x, y) \xrightarrow{x \rightarrow \infty} \tau_0 = \tau_{1,0}(y) + i\tau_{2,0}(y)$,

$$\alpha = \sqrt{-if_0} = \sqrt{2\pi} e^{-i\pi/4} = \pi - i\pi_0$$

二、化 简

显然在一般情况下求解问题(5)是很困难的,我们将只在 εy 为小参数的条件下求解该问题。为此取求解海域在 Y 方向的宽度不超过 2000 公里,即 -1000 公里 $< Y < 1000$ 公里,若取 $\phi_0 = 30^\circ$ 则 $\varepsilon y < 0.27$ 可以作为小参数。若再取 $D_h = 75$ 公里,则 $y < 13.3$, $\varepsilon = 0.02$ 。 ε 为更高阶的小量。进而我们假定风场在 y 方向的变化是很缓慢的,可以用小参数 εy 展成如下形式:

$$\tau(x, y) = \tau_0(x) + \varepsilon y \tau_1(x) + (\varepsilon y)^2 \tau_2(x) + \dots \quad (6)$$

这样在边值问题(5)中, y 只是以小参数 εy 的形式出现的。我们自然会想到将解用小参数 εy 展开:

$$\begin{aligned} W(x, y, z) &= W_0(x, z) + \varepsilon y W_1(x, z) + (\varepsilon y)^2 W_2(x, z) + \dots, \\ w(x, y, z) &= w_0(x, z) + \varepsilon y w_1(x, z) + (\varepsilon y)^2 w_2(x, z) + \dots, \\ D(x, y) &= D_0(x) + \varepsilon y D_1(x) + (\varepsilon y)^2 D_2(x) + \dots, \end{aligned} \quad (7)$$

由于 $\left(\frac{\partial \zeta}{\partial X} + \frac{\partial \zeta}{\partial Y} \right) / \zeta'_0 = D(x, y)$,

所以 $\frac{\partial}{\partial y} \text{Re}D(x, y) = \frac{\partial}{\partial x} \text{Im}D(x, y)$,

即 $\varepsilon \text{Re}[D_1(x) + 2\varepsilon y D_2(x) + 3(\varepsilon y)^2 D_3(x) + \dots] = \text{Im}[D'_0(x) + \varepsilon y D'_1(x) + (\varepsilon y)^2 D'_2(x) + \dots]$

于是

$$I_m D'_0(x) = \varepsilon \operatorname{Re} D_1(x),$$

$$I_m D'_1(x) = 2\varepsilon \operatorname{Re} D_2(x),$$

⋮

$$I_m D'_n(x) = (n+1)\varepsilon \operatorname{Re} D_{n+1}(x),$$

或

$$I_m D_n(x) = (n+1)\varepsilon \operatorname{Re} \int_0^x D_{n+1}(x) dx,$$

$$I_m \frac{D_n(x)}{h(x)} = (n+1)\varepsilon \operatorname{Re} \int_0^x D_{n+1}(x) dx / h(x)$$

$$= \varepsilon \cdot 0(D_{n+1}) \approx 0.$$

(8)

将展开式(6)和(7)代入边值问题(5),从运动方程得到

$$\begin{aligned} & \Delta W_0 + \varepsilon y \Delta W_1 + (\varepsilon y)^2 \Delta W_2 + 2\varepsilon^2 W_2 + 6\varepsilon^3 y W_3 + 12\varepsilon^4 y^2 W_4 + \dots \\ & - if_0(1 + \varepsilon y)[W_0 + \varepsilon y W_1 + (\varepsilon y)^2 W_2 + \dots] \\ & = D_0 + \varepsilon y D_1 + (\varepsilon y)^2 D_2, \end{aligned}$$

比较 y 的同幂次项,再忽略高阶小量得到

$$y^0: \Delta W_0 + 2\varepsilon^2 W_2 - if_0 W_0 = D_0,$$

$$\Delta W_0 - if_0 W_0 = D_0,$$

$$y^1: \Delta W_1 + 6\varepsilon^2 W_3 - if_0 W_0 - if_0 W_1 = D_1, \quad \Delta W_1 - if_0 W_1 = D_1 + if_0 W_0,$$

$$y^2: \Delta W_2 + 12\varepsilon^2 W_4 - if_0 W_1 - if_0 W_2 = D_2, \quad \Delta W_2 - if_0 W_2 = D_2 + if_0 W_1,$$

⋮

⋮

于是 W_n 所满足的运动方程的一般形式为

$$\Delta W_n - if_0 W_n = D_n + if_0 W_{n-1}, \quad n = 0, 1, 2, \dots \quad (9)$$

(其中 $W_{-1} = 0$)。

由连续性方程

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

可得到垂直速度

$$\begin{aligned} w = & - \int_{-h}^z \frac{\partial u}{\partial x} dz - \int_{-h}^z \frac{\partial v}{\partial y} dz = - \int_{-h}^z dz [u'_0 + \varepsilon y u'_1 + (\varepsilon y)^2 u'_2 + \dots \\ & + \varepsilon v_1 + 2\varepsilon^2 y v_2 + 3\varepsilon^3 y^2 v_3 + \dots]. \end{aligned}$$

在 y 的同幂次项中忽略高阶小量得到¹⁾

$$w = - \int_{-h}^z \frac{\partial u}{\partial x} dz, \quad (10)$$

这就是确定垂直速度的公式。

连续性方程还可写为

$$\begin{aligned} 0 = & \int_{-h}^{\zeta} \frac{\partial u}{\partial x} dz + \int_{-h}^{\zeta} \frac{\partial v}{\partial y} dz + w|_{\zeta} = \int_{-h}^{\zeta} \frac{\partial u}{\partial x} dz + \int_{-h}^{\zeta} \frac{\partial v}{\partial y} dz + \frac{d\zeta}{dt} \\ = & \int_{-h}^{\zeta} \frac{\partial u}{\partial x} dz + \int_{-h}^{\zeta} \frac{\partial v}{\partial y} dz + u|_{\zeta} \frac{\partial \zeta}{\partial x} + v|_{\zeta} \frac{\partial \zeta}{\partial y} \end{aligned}$$

1) 由于地转作用有 $U \sim V$ (同量阶)或 $u \sim v$ 。摩擦宽度 D_h 表示侧向摩擦作用范围,所以 $\frac{\partial u}{\partial x} \sim u_{D_h} \sim 0$

$$= \frac{\partial}{\partial x} \int_{-h}^{\xi} u dz + \frac{\partial}{\partial y} \int_{-h}^{\xi} v dz$$

$$\text{或} \quad \int_{-h}^{\xi} u dz = - \int_0^x dx \left[\frac{\partial}{\partial y} \int_{-h}^{\xi} v dz \right],$$

由展开式 (7) 可得

$$\frac{1}{h} \int_{-h}^0 u dz = \bar{u}_0 + \varepsilon y \bar{u}_1 + (\varepsilon y)^2 \bar{u}_2 + \dots = \frac{\varepsilon}{h} \int_0^x dx \int_{-h}^z$$

$$\times [v_1 + 2\varepsilon y v_2 + 3(\varepsilon y)^2 v_3 + \dots] dz$$

$$\text{即} \quad \bar{u}_n = (n+1)\varepsilon \frac{1}{h} \int_0^x dx \int_{-h}^z v_{n+1} dz - \varepsilon \cdot 0 \left(\int_{-h}^z v_{n+1} dz \right) \approx 0,$$

所以连续性方程的简化形式为

$$\operatorname{Re} \bar{W}_n = \operatorname{Re} \frac{1}{h} \int_{-h}^0 W_n dz = 0, \quad (11)$$

而海面边界条件变为

$$\frac{\partial W_n}{\partial z} \Big|_{z=0} = \tau_n(x). \quad (12)$$

这样我们就得到 W_n, D_n 所满足的边值问题为

$$\begin{cases} \Delta W_n - if_0 W_n = D_n + if_0 W_{n-1}, & (W_{-1} = 0) \\ \operatorname{Re} \frac{1}{h} \int_{-h}^0 W_n dz = 0, \\ \frac{\partial W_n}{\partial z} \Big|_{z=0} = \tau_n(x), \\ W_n \Big|_{x=0} = W_n \Big|_{x=-h} = 0, \\ W_n \rightarrow W_{nE} \equiv \tau_n^0 \frac{e^{i\alpha z}}{i\alpha}, \quad \text{当 } x \rightarrow \infty \\ w_n = -\operatorname{Re} \int_{-h}^z \frac{\partial W_n}{\partial x} dz, \\ \text{其中 } \tau_n^0 = \lim_{x \rightarrow \infty} \tau_n(x). \end{cases} \quad (13)$$

三、求解边值问题 (13)

将流速 $W_n(x, z)$ 分解为两部分 W_c 和 W_g , 即

$$W_n = W_{n,c} + W_{n,g}, \quad (14)$$

$$\text{其中} \quad W_{n,c} = \tau_n(x) \frac{\sin \alpha(z+h)}{\alpha \cos \alpha h} = \tau_n(x) [\sin \alpha z + \operatorname{tg} \alpha h \cos \alpha z],$$

$$\alpha = \sqrt{-if_0} = \pi - i\pi$$

满足如下关系

$$\begin{cases} \Delta W_{n,c} - if_0 W_{n,c} \equiv (W_{n,c})''_x, \\ \bar{W}_{n,c} \equiv \frac{1}{h} \int_{-h}^0 W_{n,c} dz = \frac{\tau_n(x)}{\alpha h} \left[\frac{1}{\cos \alpha h} - 1 \right], \end{cases}$$

$$\begin{cases} \left. \frac{\partial W_{n,c}}{\partial z} \right|_{z=0} = \tau_n(x), W_{n,c}|_{z=0} = W_{n,c}|_{z=-h} = 0, \\ W_{n,c} \rightarrow W_{n,E} = \tau_n^0 \frac{e^{i\alpha z}}{i\alpha} \text{ (无限深海的 Ekman 漂流)} \end{cases} \quad (15)$$

于是函数 $W_{n,g} = W_n - W_{n,c}$ 所满足的边值问题为

$$\begin{cases} \Delta W_{n,g} - if_0 W_{n,g} = D_n + if_0 W_{n-1,g} + [if_0 W_{n-1,c} - (W_{n,c})'_x], \\ \operatorname{Re} \int_{-h}^0 \frac{1}{h} W_{n,g} dz = -\operatorname{Re} \bar{W}_{n,c}, \\ \left. \frac{\partial W_{n,g}}{\partial z} \right|_{z=0} = 0, W_{n,g}|_{z=0} = W_{n,g}|_{z=-h} = 0, W_{n,g}|_{z \rightarrow \infty} = 0, \end{cases} \quad (16)$$

其中 $W_{n,c}$, $\bar{W}_{n,c}$ 分别由 (14), (15) 式给出。

四、求解边值问题(16)

为求解方便起见, 引进极坐标 (r, θ) : $x = r \cos \theta$, $z = r \sin \theta$, θ 以逆时针旋转为正。于是平均海平面为 $\theta = 0$, 海底为 $\theta = -\theta_0$ 。

取解 $W_{n,g}$ 为如下分离变量的形式

$$W_{n,g}(r, \theta) = \sum_{m=0}^{\infty} [A_m^n(r) R_m^{\otimes}(r) + \mathcal{B}_m^n R_m^{\otimes}(r)] (\cos \nu_m \theta_0 - \cos \nu_m \theta) \quad (17)$$

其中 \mathcal{B}_m^n 为常数, $A_m^{-1}(r) = \mathcal{B}_m^{-1} = 0$ 。

显然它满足边界条件:

$$\left. \frac{\partial W_{n,g}}{\partial z} \right|_{z=0} = \left. \frac{\partial W_{n,g}}{r \partial \theta} \right|_{\theta=0} = 0, W_{n,g}|_{z=-h} = W_{n,g}|_{\theta=-\theta_0} = 0. \quad (18)$$

为使 $\{\cos \nu_m \theta\}$ 在区间 $(-\theta_0, 0)$ 上构成一完备正交系, 需要取 $\nu_m = \frac{m\pi}{\theta_0}$, $m = 0, 1, 2, \dots$

先计算

$$\begin{aligned} (\Delta - if_0) AR \cos \nu \theta &= \left[\Delta_1 + \left(\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - if_0 \right) \right] AR \cos \nu \theta \\ &= (\Delta_1 AR) \cos \nu \theta + AR \left(\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - if_0 \right) \cos \nu \theta \\ &= \left(A''R + 2A'R' + \frac{1}{r} A'R \right) \cos \nu \theta + A \left[R'' + \frac{1}{r} R' \right. \\ &\quad \left. + \left(-if_0 - \frac{\nu^2}{r^2} \right) R \right] \cos \nu \theta \\ &= \frac{(rR^2A')'}{rR} \cos \nu \theta + A \left[R'' + \frac{1}{r} R' + \left(-if_0 - \frac{\nu^2}{r^2} \right) R \right] \cos \nu \theta \end{aligned} \quad (19)$$

其中 $\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$,

$$\begin{aligned} (\Delta - if_0) AR \cos \nu \theta_0 &= (\Delta_1 - if_0) AR \cos \nu \theta_0 \\ &= \left(A''R + 2A'R' + \frac{1}{r} A'R \right) \cos \nu \theta_0 + A \left[R'' + \frac{1}{r} R' - if_0 R \right] \cos \nu \theta_0 \end{aligned}$$

$$= \frac{(rR^2A')'}{rR} \cos \nu\theta_0 + A \left\{ \left[R'' + \frac{1}{r} R' + \left(-if_0 - \frac{\nu^2}{r^2} \right) R \right] + \frac{\nu^2}{r^2} R \right\} \cos \nu\theta_0. \quad (20)$$

将解式(17)代到边值问题(16)的运动方程中,利用(19),(20)式得到

$$\begin{aligned} & \sum_{m=0}^{\infty} A_m^n \left[R_m^{\oplus''} + \frac{1}{r} R_m^{\oplus'} + \left(-if_0 - \frac{\nu_m^2}{r^2} \right) R_m^{\oplus} \right] (\cos \nu_m\theta_0 - \cos \nu_m\theta) \\ & + \sum_{m=0}^{\infty} \frac{(rR_m^{\oplus 2}A_m^n)'}{rR_m^{\oplus}} (\cos \nu_m\theta_0 - \cos \nu_m\theta) + \sum_{m=0}^{\infty} \frac{\nu_m^2}{r^2} A_m^n R_m^{\oplus} \cos \nu_m\theta_0 \\ & + \sum_{m=0}^{\infty} \mathcal{B}_m^n \left[R_m^{\otimes''} + \frac{1}{r} R_m^{\otimes'} + \left(-if_0 - \frac{\nu_m^2}{r^2} \right) R_m^{\otimes} \right] (\cos \nu_m\theta_0 - \cos \nu_m\theta) \\ & + \sum_{m=0}^{\infty} \frac{\nu_m^2}{r^2} \mathcal{B}_m^n R_m^{\otimes} \cos \nu_m\theta_0 \quad (21) \\ & = D_n + [if_0 \mathbf{W}_{n-1,c} - (\mathbf{W}_{n,c})_x''] + if_0 \sum_{m=0}^{\infty} [A_m^{n-1} R_m^{\oplus} \\ & + \mathcal{B}_m^{n-1} R_m^{\otimes}] (\cos \nu_m\theta_0 - \cos \nu_m\theta), \end{aligned}$$

取 $R_m^{\oplus}, R_m^{\otimes}$ 为方程

$$R'' + \frac{1}{r} R' + \left(-if_0 - \frac{\nu_m^2}{r^2} \right) R = 0 \quad (22)$$

的两个线性无关的非零解。

再将 $[if_0 \mathbf{W}_{n-1,c} - (\mathbf{W}_{n,c})_x'']$ 展成富氏级数(见附录)

$$-if_0 \mathbf{W}_{n-1,c} + (\mathbf{W}_{n,c})_x'' = \sum_{m=0}^{\infty} B_m^n(r) \cos \nu_m\theta, \quad (23)$$

其中
$$B_m^n(r) = \frac{2}{\theta_0} \int_{-\theta_0}^0 [-if_0 \mathbf{W}_{n-1,c} + (\mathbf{W}_{n,c})_x''] \cos \nu_m\theta d\theta, \quad (23)'$$

则方程(21)变为

$$\begin{aligned} & \sum_{m=1}^{\infty} \frac{(rR_m^{\oplus 2}A_m^n)'}{rR_m^{\oplus}} (\cos \nu_m\theta_0 - \cos \nu_m\theta) + \sum_{m=1}^{\infty} \frac{\nu_m^2}{r^2} (A_m^n R_m^{\oplus} + \mathcal{B}_m^n R_m^{\otimes}) \cos \nu_m\theta_0 \\ & = D_n(r) + \sum_{m=1}^{\infty} B_m^n(r) (\cos \nu_m\theta_0 - \cos \nu_m\theta) \\ & + if_0 \sum_{m=1}^{\infty} [A_m^{n-1} R_m^{\oplus} + \mathcal{B}_m^{n-1} R_m^{\otimes}] (\cos \nu_m\theta_0 - \cos \nu_m\theta) \\ & - \sum_{m=0}^{\infty} B_m^n(r) \cos \nu_m\theta_0, \quad (24) \end{aligned}$$

其中
$$\sum_{m=0}^{\infty} B_m^n(r) \cos \nu_m\theta_0 = [-if_0 \mathbf{W}_{n-1,c} + (\mathbf{W}_{n,c})_x'']_{\theta=-\theta_0} = 0.$$

比较方程两边得到

$$\frac{(rR_m^{\oplus 2}A_m^n)'}{rR_m^{\oplus}} = B_m^n(r) + if_0 [A_m^{n-1}(r) R_m^{\oplus}(r) + \mathcal{B}_m^{n-1} R_m^{\otimes}(r)] \equiv E_m^n(r) \quad (25)$$

(其中 $A_m^{-1} = \mathcal{B}_m^{-1} = 0$),

$$D_n(r) = \sum_{m=1}^{\infty} \frac{\nu_m^2}{r^2} [A_m^n(r) R_m^{\circledast}(r) + \mathcal{B}_m^n R_m^{\circledast}(r)] \cos \nu_m \theta_0, \quad (26)$$

由(25)式可解出

$$A_m^n(r) = \int_{r_1}^r \frac{d\eta}{\eta R_m^{\circledast}(\eta)} \int_{r_0}^{\eta} \xi R_m^{\circledast}(\xi) E_m^n(\xi) d\xi + \mathcal{A}_m^n \int_{r_2}^r \frac{d\eta}{\eta R_m^{\circledast}(\eta)} + C_m^n. \quad (27)$$

五、求基本函数 R_m^{\circledast} , R_m^{\circledast}

方程(22)是复变量的 Bessel 方程,事实上引进新变量

$$\xi = \sqrt{if_0} r = \sqrt{2} \pi e^{i\pi/4} r = (\pi + i\pi)r \quad (28)$$

后,方程(22)变为^[1]:

$$\frac{d^2 R}{d\xi^2} + \frac{1}{\xi} \frac{dR}{d\xi} - \left(1 + \frac{\nu^2}{\xi}\right) R = 0. \quad (29)$$

该方程的两个线性无关解是第一类变型 Bessel 函数

$$I_\nu(\xi) = \left(\frac{\xi}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{1}{l!} \frac{1}{\Gamma(\nu + l + 1)} \left(\frac{\xi}{2}\right)^{2l} \quad (30)$$

和第二类变型 Bessel 函数

$$K_\nu(\xi) = \frac{\pi}{2 \sin \nu\pi} [I_{-\nu}(\xi) - I_\nu(\xi)], \quad (31)$$

我们取这两组 Bessel 函数作为构造解(17)的基本函数。

当 $\nu > 0$ 时, $\xi = 0$ 是函数 $I_\nu(\xi)$ 的零点和函数 $I_{-\nu}(\xi)$ 的奇点,因而也是函数 $K_\nu(\xi)$ 的奇点。由于 $|\xi| \rightarrow \infty$ 时函数 $I_\nu(\xi)$ 和 $K_\nu(\xi)$ 的渐近式分别为:

$$I_\nu(\xi) = \frac{e^\xi}{\sqrt{2\pi\xi}} \sum_{l=0}^{\infty} \frac{(-)^l (\nu, l)}{(2\xi)^l} + \frac{e^{-\xi + (\nu+1/2)\pi i}}{\sqrt{2\pi\xi}} \sum_{l=0}^{\infty} \frac{(\nu, l)}{(2\xi)^l} \quad (32)$$

$$(-\pi/2 < \arg \xi < 3\pi/2)$$

和

$$K_\nu(\xi) = \sqrt{\frac{\pi}{2\xi}} e^{-\xi} \left[1 + \sum_{l=1}^{\infty} \frac{(\nu, l)}{(2\xi)^l}\right], \quad |\arg \xi| < \frac{3\pi}{2}. \quad (33)$$

所以 $r \rightarrow \infty$ 是 $I_\nu(\xi)$ 的奇点, $K_\nu(\xi)$ 的零点。

六、构造解 $W_{n,g}$

由于 Bessel 函数 I_ν , K_ν 都不能在 $r = 0$ 和 $r = \infty$ 处同时保持正则性质,为了构造一个满足 $W_{n,g}|_{r=0} = W_{n,g}|_{r=\infty} = 0$ 的解,我们将采用分段解,然后再将两部分解匹配起来。

i) 在 $r = 0$ 的邻域 $r < r_0$, 我们取

$$W_{n,g} = W_{n,g}^{\circledast} = \sum_{m=1}^{\infty} \{ [P_m^{\circledast}(r) + \mathcal{A}_m^{n,\circledast} Q_m^{\circledast}(r)] K_{\nu_m}(\xi) + \mathcal{B}_m^{n,\circledast} I_{\nu_m}(\xi) \} [\cos \nu_m \theta_0 - \cos \nu_m \theta] \quad (34)$$

其中

$$P_m^{n,0}(r) = \int_0^r \frac{dr}{r K_{\nu_m}^2(\xi)} \int_{r_0}^r r K_{\nu_m}(\xi) E_m^n(r) dr,$$

$$Q_m^{0}(r) = \int_0^r \frac{dr}{r K_{\nu_m}^2(\xi)}, \quad \mathcal{A}_m^{n,0} \text{ 和 } \mathcal{B}_m^{n,0} \text{ 为复常数。}$$

ii) 在 $r = \infty$ 的邻域 $r > r_0$, 我们取

$$W_{n,g} = W_{n,g}^{0} = \sum_{m=1}^{\infty} \{ [P_m^{n,0}(r) + \mathcal{A}_m^{n,0} Q_m^{0}(r)] I_{\nu_m}(\xi) + \mathcal{B}_m^{n,0} K_{\nu_m}(\xi) \} [\cos \nu_m \theta_0 - \cos \nu_m \theta] \quad (35)$$

其中

$$P_m^{n,0}(r) = \int_{\infty}^r \frac{dr}{r I_{\nu_m}^2(\xi)} \int_{r_0}^r r I_{\nu_m}(\xi) E_m^n(r) dr,$$

$$Q_m^{0}(r) = \int_{\infty}^r \frac{dr}{r I_{\nu_m}^2(\xi)}, \quad \mathcal{A}_m^{n,0} \text{ 和 } \mathcal{B}_m^{n,0} \text{ 为复常数。}$$

iii) 在 $r = r_0$ 的断面上将两部分解匹配起来 (见图 2) 即要求函数本身和其对 r 的一阶微商, 二阶微商在 $r = r_0$ 处连续:

$$W_{n,g}^{0}(r_0, \theta) = W_{n,g}^{0}(r_0, \theta), \quad (36)$$

$$\frac{\partial W_{n,g}^{0}(r_0, \theta)}{\partial r} = \frac{\partial W_{n,g}^{0}(r_0, \theta)}{\partial r}, \quad (37)$$

$$\frac{\partial^2 W_{n,g}^{0}(r_0, \theta)}{\partial r^2} = \frac{\partial^2 W_{n,g}^{0}(r_0, \theta)}{\partial r^2}. \quad (38)$$

iv) 上面选取的分段解中包含 4 组复常数

$\{ \mathcal{A}_m^{n,0}, \mathcal{B}_m^{n,0}, \mathcal{A}_m^{n,0}, \mathcal{B}_m^{n,0} \}$, 它们可以由匹配条件 (36), (37), (38) 和问题 (16) 中的连续性方程

$$\operatorname{Re} \frac{1}{h} \int_{-h}^0 W_{n,g} dz = -\operatorname{Re} \bar{W}_{n,e} \quad 0 < x < \infty \quad (39)$$

和条件 (8)

$$I_m \frac{D_n(x)}{h(x)} = 0 \quad (8)$$

唯一确定。据 (26), (34), (35) 式, 其中

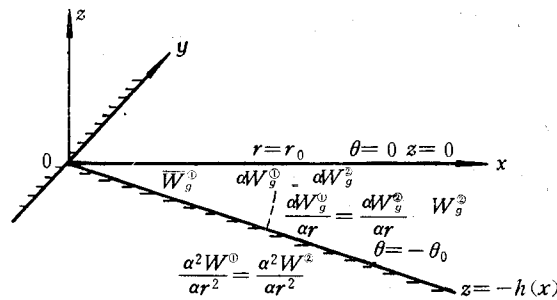


图 2 分段解适用范围

$$D_n(x) = D_n(r) = \sum_{m=1}^{\infty} \frac{(-1)^m \nu_m^2}{r^2} \{ [P_m^{n, \textcircled{0}}(r) + \mathcal{A}_m^{n, \textcircled{0}} Q_m^{\textcircled{0}}(r)] K_{\nu_m}(\xi) + \mathcal{B}_m^{n, \textcircled{0}} I_{\nu_m}(\xi) \}, \text{ 当 } r < r_0, \quad (40)$$

和

$$D_n(x) = D_n(r) = \sum_{m=1}^{\infty} \frac{(-1)^m \nu_m^2}{r^2} \{ [P_m^{n, \textcircled{0}}(r) + \mathcal{A}_m^{n, \textcircled{0}} Q_m^{\textcircled{0}}(r)] I_{\nu_m}(\xi) + \mathcal{B}_m^{n, \textcircled{0}} K_{\nu_m}(\xi) \}, \text{ 当 } r_0 < r. \quad (41)$$

v) 可以证明这样确定的解 W_g (省去脚标 n) 是满足 $r = 0$ 和 $r = \infty$ 处的边界条件的

$$W_g^{\textcircled{0}}|_{r=0} = W_g^{\textcircled{0}}|_{r=\infty} = 0. \quad (42)$$

事实上

1) 当 $r \rightarrow 0$ 时, 据表达式 (30), (31) 和附录可知

$$K_\nu(\xi) = O(r^{-\nu}), \quad I_\nu(\xi) = O(r^\nu), \quad E(r) = O(1),$$

所以

$$\begin{aligned} |P^{\textcircled{0}} K_\nu| &\sim r^{-\nu} \int_0^r \frac{dr}{r \cdot r^{-2\nu}} \int_{r_0}^r r \cdot r^{-\nu} dr = r^{-\nu} \int_0^r r^{2\nu-1} dr \int_{r_0}^r r^{1-\nu} dr \\ &\sim r^{-\nu} \int_0^r r^{2\nu-1} (r^{2-\nu} - r_0^{2-\nu}) dr \sim O(r^2), \end{aligned}$$

$$|Q^{\textcircled{0}} K_\nu| \sim r^{-\nu} \int_0^r \frac{dr}{r \cdot r^{-2\nu}} = r^{-\nu} \int_0^r r^{2\nu-1} dr \sim O(r^\nu), \quad (\nu \geq 2)$$

总有 $W_g^{\textcircled{0}} = O(r^2)$ 当 $r \rightarrow 0$ 。

2) 当 $r \rightarrow \infty$ 时, 由渐近式 (32), (33) 和附录可知

$$K_\nu(\xi) = O\left(\frac{e^{-\sqrt{2}\pi r}}{\sqrt{r}}\right), \quad I_\nu(\xi) = O\left(\frac{e^{\sqrt{2}\pi r}}{\sqrt{r}}\right) \text{ 和 } E(r) = O(r^{-1}),$$

所以

$$\begin{aligned} |P^{\textcircled{0}} I_\nu| &\sim \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r \frac{dr}{r} \int_{r_0}^r r \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} r^{-1} dr \\ &= \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r \frac{\sqrt{\eta} d\eta}{e^{2\sqrt{2}\pi r}} \int_{r_0}^{\eta} \frac{e^{\sqrt{2}\pi \xi}}{\sqrt{\eta} \sqrt{\xi}} d\xi < \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r \sqrt{\eta} e^{-2\sqrt{2}\pi \eta} d\eta \int_{r_0}^{\eta} \frac{e^{\sqrt{2}\pi \xi}}{\xi} d\xi \\ &\stackrel{[3]}{\sim} \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r \sqrt{\eta} e^{-2\sqrt{2}\pi \eta} \frac{e^{\sqrt{2}\pi \eta}}{\eta} d\eta = \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r \frac{e^{-\sqrt{2}\pi \eta}}{\sqrt{\eta}} d\eta \\ &\stackrel{[3]}{\sim} \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \cdot \frac{e^{-\sqrt{2}\pi r}}{\sqrt{r}} = O(r^{-1}), \\ |Q^{\textcircled{0}} I_\nu| &\sim \frac{e^{\sqrt{2}\pi r}}{\sqrt{r}} \int_0^r e^{-2\sqrt{2}\pi r} dr = O\left(\frac{e^{-\sqrt{2}\pi r}}{\sqrt{r}}\right), \end{aligned}$$

总有 $W_g^{\textcircled{0}} = O(r^{-1})$, 当 $r \rightarrow \infty$ 。

至此我们已原则上完成了解的构造工作, 它满足了所有需要满足的方程和边界条件。余下的具体计算工作将不在本文中进行。

附 录

这里我们将给出计算 $B_m^0(r)$ [见 (23)' 式] 的方法, 并估计当 $r \rightarrow 0$ 和 $r \rightarrow \infty$ 时 $B(r)$ (省去角标) 的量阶和 $E(r)$ [即 (25) 式的 $E_m^0(r)$] 的量阶。

将 (14) 式中 W_0 的表达式代入 (23)' 中, 可以看出问题的关键是计算积分

$$S_1 = \int_0^{\theta_0} \sin \alpha x \cos \nu_m \theta d\theta \quad \text{和} \quad S_2 = \int_0^{\theta_0} \cos \alpha x \cos \nu_m \theta d\theta。$$

现试计算 S_1 , 实际上

$$\begin{aligned} S_1 &= \int_0^{\theta_0} \sin(\alpha r \sin \theta) \cos \nu_m \theta d\theta = \int_0^{\theta_0} \sin(\alpha r \sin \theta) \frac{\cos \nu_m \theta}{\cos \theta} d(\sin \theta) \\ &= \int_0^{\sin \theta_0} \sin \alpha r \eta \frac{\cos \nu_m \theta}{\cos \theta} d\eta = \int_0^{\eta_0 = \sin \theta_0} \sin \alpha r \eta f(\eta) d\eta, \end{aligned}$$

其中

$$f(\eta) = \frac{\cos \nu_m \theta}{\cos \theta}, \quad \theta = \theta(\eta), \quad \eta = \sin \theta。$$

在 $\theta \in (0, \theta_0)$, $\theta_0 < \pi/2$, $f(\eta)$ 是 η 的解析函数, 可以展成一致收敛的幂级数, 并且可以用前有限项和来逼近 $f(\eta)$, 即

$$f(\eta) \doteq \sum_{n=0}^N a_n \eta^n。$$

于是

$$\begin{aligned} S_1 &= \int_0^{\eta_0} \sin \alpha r \eta \sum_{n=0}^N a_n \eta^n d\eta = \frac{-\cos \alpha r \eta}{\alpha r} \sum_{n=0}^N a_n \eta^n \Big|_0^{\eta_0} \\ &\quad + \frac{1}{\alpha r} \int_0^{\eta_0} \cos \alpha r \eta \sum_{n=1}^N n a_n \eta^{n-1} d\eta。 \end{aligned}$$

如此分部积分 N 次就可得到最终结果, 并且可以看出, 当 $r \rightarrow \infty$ 时 $S_1 = O(r^{-1})$ 。对 S_2 也可同样处理, 并有同样的量阶估计。经过一些计算, 根据 S_1 和 S_2 的量阶估计, 可以证明, 当 $r \rightarrow \infty$ 时 $B(r) = O(r^{-1})$, 并且只要 $\mathbf{x}(x, y)$ 在 $x = 0$ 附近足够光滑, $B(r)$ 在 $r = 0$ 附近就是正则的, 即当 $r \rightarrow 0$ 时 $B(r) = O(1)$ 。

据 (25) 式 $E_m^0(r) = B_m^0(r)$, 所以 $E_m^0(r) = O(1)$ (当 $r \rightarrow 0$) 和 $E_m^0(r) = O(r^{-1})$ (当 $r \rightarrow \infty$)。而据第五部分 v) 中的推导可知, 从 $r \rightarrow 0$ 时 $E^*(r) = O(1)$ 就可推得 $E^{*+1}(r) = B^{*+1}(r) + O(r^2) = O(1)$; 从 $r \rightarrow \infty$ 时 $E^*(r) = O(r^{-1})$ 就可推得 $E^{*+1}(r) = B^{*+1}(r) + O(r^{-1}) = O(r^{-1})$ 。所以据数学归纳法, 我们总有量阶估计 $E(r) = O(1)$ (当 $r \rightarrow 0$) 和 $E(r) = O(r^{-1})$ (当 $r \rightarrow \infty$)。

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A MODEL FOR CALCULATING WIND-DRIVEN OCEAN CURRENT ON CONTINENTAL SHELF*

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ABSTRACT

In studying wind-driven ocean current on continental shelf, the sea surface slope and the bottom topography must be taken into account, making it difficult to find an analytical solution.

In this paper, the problem of a steady current in a sea with a linear sloping bottom under a two-dimensional steady wind condition is discussed. By perturbation method, a series of systems of equations can be obtained by means of expanding the solution and wind-stress (if is possible) in a series of small parameter βy (β -effect).

Taking Bessel functions as basic functions, a series-type solution on the polar coordinate system with separable variables was obtained. In order to keep the solution regular at a tip-point near coast, we first found out two section-type solutions, then "matched" them at some distance from the boundary, and finally, a uniform effective solution for the whole region was obtained. It appears that by this method the difficulty R. B. Hill and J. A. Johnson met in their paper entitled "A three-dimensional theory of coastal currents and upwelling over a continental shelf" can be overcome.

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